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Math 565: Functional Analysis | HOMEWORK 3 | **Due: Feb 19, 23:59**

Definition. In a real vector space X, a set $U \subseteq X$ is called **convex** if it is closed under convex combinations, i.e. $\alpha x + (1-\alpha)y \in U$ for all $x, y \in U$ and $\alpha \in [0,1]$.

1. Let X be a real normed vector space and $U \subseteq X$ be a convex open set with $0 \in U$. Define the **Minkowski functional** $p : X \to \mathbb{R}$ of U (also called the **gauge** of U) by

$$p(x) := \inf \left\{ \alpha > 0 : \alpha^{-1} x \in U \right\}.$$

Prove:

(a) p is a nonnegative sublinear functional.

HINT: For subadditivity, note that $(p(x)+\varepsilon)^{-1}x \in U$ for each $x \in X$ and each $\varepsilon > 0$, and use that $(\alpha+\beta)^{-1}(x+y)$ is a convex combination of $\alpha^{-1}x$ and $\beta^{-1}y$ for all $x, y \in X$ and $\alpha, \beta > 0$.

(b) p is not a semi-norm in general by providing a counterexample in $X := \mathbb{R}$.

(c) p is a bounded functional, i.e. there is a constant $C \geqslant 0$ such that $p(x) \leqslant C\|x\|$ for all $x \in X$.

(d) $U = \{p < 1\}$.

2. Let X be a real normed vector space and $U \subseteq X$ be a convex open set. Prove that for each $x_0 \in X \setminus U$, there is $f \in X^*$ such that $f(u) < f(x_0)$ for all $u \in U$.

RESTRICTIONS: This statement is part of the proof of geometric Hahn–Banach, so you may not use the latter in your solution. You may use ordinary Hahn–Banach (on extension of functionals).

HINT: Translating U and x_0 by the same vector, we may assume $0 \in U$, so $x_0 \neq 0$. Start with $f(x_0) := 1$ and extend it to X via the Minkowski functional.

3. Let X and Y be normed vector spaces and $T \in B(X,Y)$. Define the **transpose** $T^t : Y^* \to X^*$ of T by $T^t f = f \circ T$.

(a) Observe that $\|T^t\| \leqslant \|T\|$, so $T^t \in B(Y^*, X^*)$.

(b) Identifying X with $\widehat{X}$ and Y with $\widehat{Y}$, check that $T^{tt}|_{\widehat{X}} = T$. Deduce that $\|T^t\| = \|T\|$.

(c) Verify that if $T(X)$ is dense in Y, then T^t is injective. Deduce that if $T^t(Y^*)$ is dense in X^*, then T is injective.

Definition. Let X be a normed vector space. We stated in class that the map $x \mapsto \widehat{x} : X \to X^{**}$ is an isometry and we denoted its image by $\widehat{X}$. Call X **reflexive** if $\widehat{X} = X^{**}$.

4. Suppose that X is a Banach space and let $Y \subseteq X$ and $F \subseteq X^*$. Put

$$Y^0 := \{f \in X^* : f|_Y = 0\} \quad \text{and} \quad F^\perp := \{x \in X : f(x) = 0 \text{ for all } f \in F\}.$$

Prove:

(a) $Y^0 = (\overline{\operatorname{Span} Y})^0$ and $F^\perp = (\overline{\operatorname{Span} F})^\perp$ and that $Y^0 = (\widehat{Y})^\perp$ and $\widehat{(F^\perp)} = F^0 \cap \widehat{X}$.

(b) $F^\perp$ is a closed subspace of X, and therefore Y^0 is a closed subspace of X^*.

(c) $(Y^0)^\perp = \overline{\operatorname{Span} Y}$ and $(F^\perp)^0 \supseteq \overline{\operatorname{Span} F}$. If X is reflexive, $(F^\perp)^0 = \overline{\operatorname{Span} F}$.

5. Let X be a Banach space.

(a) Prove that if X^* is separable, then X is separable.

HINT: Let F be a countable dense subset of X^* and for each $f \in F$, choose a unit vector $x_f \in X$ with $|f(x_f)| \geqslant \frac{1}{2}\|f\|$. Then the span of $\{x_f : f \in F\}$ is dense in X.

(b) Deduce that if X is reflexive, then the converse holds as well: if X separable then X^* is separable.

(c) Give an example of a separable Banach space X with a non-separable dual X^*.

6. Let $c := c(\mathbb{N})$ be the space of all convergent sequences in $\mathbb{C}$ and let $f \in c^*$ be a positive linear functional on c which is shift-invariant (i.e. $f \circ s = f$, where s is the shift transformation) and satisfies $\|f\| = 1$ and $f(\mathbb{1}) = 1$. Prove that f is the limit functional, i.e. $f(x) = \lim_{n\to\infty} x(n)$.

HINT: The space c_c of all eventually 0 sequences in $\mathbb{C}$ is contained in $\ker f$.

7. Let X be a normed vector space and $Y \subseteq X$ be a closed subspace. Show that $(X/Y)^* \cong Y^0$; more precisely, prove that the map $f \mapsto f \circ \pi : (X/Y)^* \to Y^0$ is an isometric isomorphism, where $\pi : X \to X/Y$ is the quotient map.

HINT: To show $\|f \circ \pi\| \geqslant \|f\|$, take a sequence $x_n + Y \in X/Y$ with $\|x_n + Y\| < 1$ and $|(f \circ \pi)(x_n)| = |f(x_n + Y)| \to \|f\|$ as $n \to \infty$. Choose $y_n \in Y$ with $\|x_n + y_n\| < 1$, so $\|f \circ \pi\| \geqslant |(f \circ \pi)(x_n + y_n)|$.

8. Let μ be an ultrafilter-measure on $\mathscr{P}(\mathbb{N})$, i.e. a $0/1$-valued finitely additive measure on $\mathscr{P}(\mathbb{N})$.

(a) Prove that for every $x \in \ell^\infty$, the limit of x along μ exists, and that $L_\mu(x) := \lim_{n\to\mu} x(n)$ is a positive linear functional on ℓ^∞.

(b) Suppose that μ is not a Dirac measure (i.e. is a nonprincipal ultrafilter). Show that every $x \in \ell^\infty$ admits a subsequence $(x(n_k))_k$ such that

$$L_\mu(x) = \lim_{k\to\infty} x(n_k),$$

where the limit on the right is the usual limit (along the Fréchet filter). Conclude that $L_\mu \in (\ell^\infty)^*$ with $\|L_\mu\| = 1$.

(c) Suppose that μ is not a Dirac measure (i.e. is a nonprincipal ultrafilter). Prove that $\widetilde{L} : \ell^\infty \to \mathbb{C}$ defined by

$$\widetilde{L}(x) := \lim_{n\to\mu} \frac{1}{n+1} \sum_{i=0}^{n} x(i)$$

is a mean on ℓ^∞.

9. [*Optional*] Given a finitely additive finite measure μ on $\mathscr{P}(\mathbb{N})$, define its integral on ℓ^∞, i.e. a positive linear functional I_μ on ℓ^∞ with $\|I_\mu\| = 1$ such that $I_\mu(\mathbb{1}_A) = \mu(A)$ for each $A \subseteq \mathbb{N}$. Observe that if μ is a $0/1$-valued measure, then $I_\mu = L_\mu$ (as in Question 8).

HINT: First define I_μ on simple functions, then extend it to non-negative functions the same way as it is done for countably additive measures.